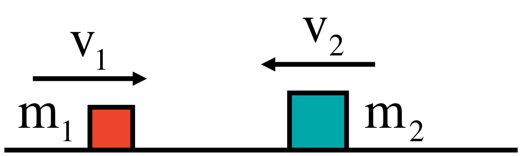
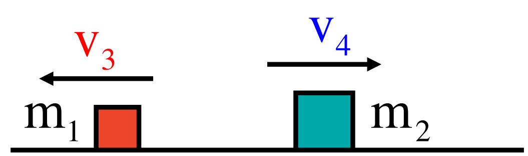
Solving v and v^2 problems

Two comparably sized masses moving in opposite directions collide elastically. The equations you can write for the collision are shown below. The question is, "How do you solve a mess like this for the "final" velocities (to keep things easy, I'll color the unknowns)?





$$m_1 v_1 - m_2 v_2 = -m_1 v_3 + m_2 v_4$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_3^2 + \frac{1}{2} m_2 v_4^2$$

The temptation is to use the momentum equations to solve for one of the variables in terms of the other, then substitute that expression into the energy equation and solve. Let's try it.

$$m_1v_1 - m_2v_2 = -m_1v_3 + m_2v_4$$

$$\Rightarrow \quad v_3 = \frac{(m_1v_1 - m_2v_2) - m_2v_4}{m_1}$$

So that:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_3^2 + \frac{1}{2}m_2v_4^2$$

$$\Rightarrow \quad \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1\left[\frac{(m_1v_1 - m_2v_2) - m_2v_4}{m_1}\right]^2 + \frac{1}{2}m_2v_4^2$$

Nasty!

So that was horrifically hard to do. There is another way, though. To do it, we need to group the velocity terms around the masses.

$$m_1 v_1 - m_2 v_2 = -m_1 v_3 + m_2 v_4$$
$$\Rightarrow \quad m_1 \left(v_1 + v_3 \right) = m_2 \left(v_2 + v_4 \right) \qquad \text{A}$$

$$\frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2 = \frac{\overset{v_1}{1}}{2} m_1 {v_3}^2 + \frac{1}{2} m_2 {v_4}^2$$
$$\Rightarrow \quad m_1 \left({v_1}^2 - {v_3}^2 \right) = m_2 \left({v_4}^2 - {v_2}^2 \right)$$

Factoring the energy expression yields:

$$m_1 \left(v_1 - v_3 \right)\left(v_1 + v_3 \right) = m_2 \left(v_4 - v_2 \right)\left(v_4 + v_2 \right) \qquad \text{B}$$

Dividing the momentum expression (A) into the modified energy expression (B) allows us to cancel a whole lot of stuff. That is:

$$\frac{m_1(v_1 - v_3)(v_1 + v_3)}{m_1(v_1 + v_3)} = \frac{m_2(v_4 - v_2)(v_4 + v_2)}{m_2(v_2 + v_4)}$$

$$\Rightarrow \quad \frac{\cancel{m_1}(v_1 - v_3)\cancel{(v_1 + v_3)}}{\cancel{m_1}\cancel{(v_1 + v_3)}} = \frac{\cancel{m_2}(v_4 - v_2)\cancel{(v_2 + v_4)}}{\cancel{m_2}\cancel{(v_2 + v_4)}}$$

$$\Rightarrow \quad (v_1 - v_3) = (v_4 - v_2)$$

$$\Rightarrow \quad v_3 = (+v_1 - v_2) - v_4$$

We now have a new, linear relationship between the two velocities. Putting it into the momentum expression yields:

We now have a new, linear relationship between the two velocities. Putting it into the momentum expression yields:

$$m_1v_1 - m_2v_2 = -m_1{\color{red}v_3} + m_2{\color{blue}v_4}$$

$$\Rightarrow \quad m_1v_1 - m_2v_2 = -m_1\left[\left(+v_1 - v_2\right) - {\color{blue}v_4}\right] + m_2{\color{blue}v_4}$$

$$\Rightarrow \quad m_1v_1 - m_2v_2 + m_1v_1 - m_1v_2 = \left(m_2 + m_1\right){\color{blue}v_4}$$

$$\Rightarrow \quad \frac{m_1v_1 - m_2v_2 + m_1v_1 - m_1v_2}{m_2 + m_1} = {\color{blue}v_4}$$

With a numerical solution for this unknown velocity, you can go back to the original momentum equation and determine the other unknown velocity.